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Quasi-local first law of black-hole dynamics

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Abstract

A property well known as the first law of black hole is a relation among infinitesimal variations of parameters of stationary black holes. We consider a dynamical version of the first law, which may be called the first law of black hole dynamics. The first law of black hole dynamics is derived without assuming any symmetry or any asymptotic conditions. In the derivation, a definition of dynamical surface gravity is proposed. In spherical symmetry it reduces to that defined recently by one of the authors (SAH).

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Black hole thermodynamics, analogies between the theory of black holes and thermodynamics, has been one of the hottest fields in black hole physics since Bekenstein's introduction of black hole entropy [1]. The black hole entropy was introduced as a quantity proportional to horizon area. The proportionality coefficient was fixed by Hawking's discovery that a black hole with surface gravity κ emits radiation with temperature $T_H = \kappa/2\pi$ [2]: by identifying T_H with the temperature of the black hole, black hole entropy is determined to be one quarter of the horizon area. The expression of black hole entropy is called the Bekenstein-Hawking formula.

To determine the coefficient in the Bekenstein-Hawking formula from the expression of the Hawking temperature T_H, the first law of black holes [3] is used. The first law is a relation among infinitesimal variations of parameters of stationary black holes: horizon area, mass, angular momentum, etc. Strictly speaking, it does not relate dynamical evolutions of these quantities. Thus, it might be physically non-trivial to connect the temperature of dynamically emitted radiation with black hole entropy by using the first law. Nonetheless, the Bekenstein-Hawking formula obtained by using the first law is acceptable. For example, in Euclidean gravity the Bekenstein-Hawking formula for a Schwarzschild black hole is correctly obtained by requiring a regularity of the corresponding Euclidean section [7].

Moreover, the first law is used in (quasi-stationary but) dynamical situations to prove the generalized second law [4,5], which is a natural generalization of both the second law (or area law [6]) of black holes and the second law of usual thermodynamics. In the proof, by assuming quasi-stationarity, the use of the first law is extended to relate small changes of physical quantities from an initial near-stationary black hole to a final near-stationary one. However, this idea of quasi-stationarity is an approximation; if we intend to prove the generalized second law for finite changes between initial and final near-stationary black holes or to a purely dynamical situation, the stationary first law can not be used.

Therefore, to make black hole thermodynamics self-consistent it must be shown that a dynamical version of the first law of black holes exists. In Ref. [8], it was derived assuming spherical symmetry and may be called the first law of black hole dynamics. The purpose of this paper is to derive the first law of black hole dynamics *without assuming any symmetry or any asymptotic conditions*.

In this paper we treat a dynamical and not necessarily asymptotically flat spacetime. Even for such a general situation, there is a definition of a black hole as a certain type of trapping horizon [9]. A *trapping horizon* is a three-surface foliated by marginal surfaces, where a *marginal surface* is a spatial two-surface on which one null normal expansion defined below vanishes. Geometrically this is where a light wave would have instantaneously parallel rays. The physical idea is that gravity can trap an expanding light wave and make it contract. We mention here that different types of trapping horizon can be regarded as defining a black hole, a white hole or a wormhole [10]. However, the distinctions are irrelevant for the purpose of this paper. The first law we shall obtain holds for any trapping horizon.

To investigate the behavior of the trapping horizon, the so-called double-null formalism or (2+2) decomposition of general relativity is useful. Among several (2+2)-formalisms [11,12], we adopt one based on Lie derivatives w.r.t null vectors developed by one of the authors [12]. Let us review basic ingredients of the formalism. Suppose that a four-dimensional spacetime manifold (M, g) is foliated (at least locally) by null hypersurfaces $\Sigma^{\pm}$, each of which is parameterized by a scalar $\xi^{\pm}$, respectively. The null character is described by $g^{-1}(n^{\pm}, n^{\pm}) = 0$, where $n^{\pm} = -d\xi^{\pm}$ are normal 1-forms to $\Sigma^{\pm}$. The relative normalization of the null normals defines a function f as $g^{-1}(n^{+}, n^{-}) = -e^{f}$. The intersections of $\Sigma^{+}(\xi^{+})$ and $\Sigma^{-}(\xi^{-})$ define a two-parameter family of two-dimensional spacelike surfaces $S(\xi^{+}, \xi^{-})$. Hence, by introducing an intrinsic coordinate system (θ^1,θ^2) of the 2-surfaces, the foliation is described by the imbedding $x = x(\xi^{+}, \xi^{-}; \theta^1, \theta^2)$.

For the imbedding, the intrinsic metric on the 2-surfaces is found to be $h = g + e^{-f}(n^{+} \otimes n^{-} + n^{-} \otimes n^{+})$. Correspondingly, the vectors $u_{\pm} = \partial/\partial\xi^{\pm}$ have 'shift vectors' $s_{\pm} = \perp u_{\pm}$, where $\perp$ indicates projection by h. The 4-dimensional metric is written in terms of (h,f,$s_{\pm}$) as

$$g = \begin{pmatrix} h(s_+, s_+) & h(s_+, s_-) - e^{-f} & h(s_+) \\ h(s_-, s_+) - e^{-f} & h(s_-, s_-) & h(s_-) \\ h(s_+) & h(s_-) & h \end{pmatrix}. \tag{1}$$

Geometrical quantities such as *expansions* $\theta_\pm$, *shears* $\sigma_\pm$ and the *twist* ω are defined by $\theta_\pm = *\mathcal{L}_\pm * 1$, $\sigma_\pm = \perp \mathcal{L}_\pm h - \theta_\pm h$ and $\omega = e^f [l_-, l_+]/2$, where $*$ denotes the Hodge-dual operator of h, $l_\pm = u_\pm - s_\pm = e^{-f} g^{-1}(n^\mp)$ are null normal vectors to $\Sigma^\pm$, and $\mathcal{L}_\pm$ denotes the Lie derivative along $l_\pm$, respectively. It is possible to write down the Einstein tensor in terms of these geometrical quantities. The component useful for our purpose is $G_{+-} = G(l_+, l_-)$, which is given by

$$2e^f G_{+-} = {}^{(2)}R + e^f(\mathcal{L}_+\theta_- + \mathcal{L}_-\theta_+ + 2\theta_+\theta_-) - 2\left[h(\omega, \omega) + \frac{1}{4}h^\sharp(df, df)\right] + \mathcal{D}^2 f. \tag{2}$$

Here $h^\sharp = g^{-1} h g^{-1}$ is h raised by g^{-1}, $\mathcal{D}^2$ and ${}^{(2)}R$ are the two-dimensional Laplacian and the Ricci scalar associated with the metric h.

Before deriving the first law we have to define energy and surface gravity in a quasi-local way. In spherical symmetry there is a widely accepted energy: the Misner-Sharp (MS) energy [13]. In Ref. [8] the MS energy is used to derive the first law of black hole dynamics in spherical symmetry. In this paper we adopt the Hawking energy [14], which reduces to the MS energy in spherical symmetry. It is defined by

$$E(\xi^+, \xi^-) = \frac{r}{16\pi} \int_{S(\xi^+, \xi^-)} d^2\theta \sqrt{h} \left[{}^{(2)}R + e^f \theta_+ \theta_-\right], \tag{3}$$

where h is the determinant of the two-dimensional metric h_{ab} and the area radius r is defined by

$$r = \sqrt{A/4\pi}, A = \int_{S(\xi^+, \xi^-)} d^2\theta \sqrt{h}. \tag{4}$$

In Ref. [8], a definition of dynamical surface gravity was proposed in spherical symmetry. A natural generalization to a not necessarily spherically symmetric case is

$$\kappa(\xi^+, \xi^-) = \frac{-1}{16\pi r} \int_{S(\xi^+, \xi^-)} d^2\theta \sqrt{h}\ e^f(\mathcal{L}_+\theta_- + \mathcal{L}_-\theta_+ + \theta_+\theta_-). \tag{5}$$

This is the most simple generalization in the sense that it includes neither the shear σ_{Aab} nor the twist ω^a.

We now derive the first law of black hole dynamics for the Hawking energy (3) and the surface gravity defined by (5). It is easy to show that

$$dE - \frac{\kappa}{8\pi}dA = wAdr + rd\left(\frac{E}{r}\right), \tag{6}$$

where w is defined by

$$w = \frac{1}{A}\left(\frac{E}{r} - \kappa r\right). \tag{7}$$

Here note that 'd' in Eq. (6) is not a variation in a space of stationary solutions of the Einstein equation as in the first law of black hole statistics, but is the differentiation w.r.t. the parameters $\xi^{\pm}$ of the spacetime foliation. (For example, $dE = d\xi^{+}\partial_{+}E + d\xi^{-}\partial_{-}E$.) We mention that Eq. (6) holds independently of the definitions of E and κ while the following arguments depend on the definitions.

A marginal surface is a surface where one of the expansions $\theta_{\pm}$ vanishes. Since the Gauss-Bonnet theorem says that

$$\int_{S(\xi^{+},\xi^{-})} d^2\theta\sqrt{h}\ {}^{(2)}R = 8\pi(1-\gamma), \tag{8}$$

where γ is the genus or number of handles of $S(\xi^{+},\xi^{-})$, the energy divided by area radius is given by $E/r = (1-\gamma)/2$ on a marginal surface and is a constant. Thus,

$$E' = \frac{\kappa}{8\pi}A' + wAr', \tag{9}$$

where the prime denotes the derivative along the trapping horizon. This is the first law of black hole dynamics. Note that this also holds along any hypersurface foliated by 2-surfaces on which E/r is constant.

By using Eq. (2) it is easy to show that w is written as follows.

$$w = w_m + w_j, \tag{10}$$

where the averaged matter energy density w_m and the effective angular energy density w_j are defined by

$$w_m = \frac{1}{8\pi A}\int_{S(\xi^+,\xi^-)} d^2\theta\sqrt{h}e^f G_{+-},$$
$$w_j = \frac{1}{8\pi A}\int_{S(\xi^+,\xi^-)} d^2\theta\sqrt{h}\left[h(\omega,\omega)+\frac{1}{4}h^\sharp(df,df)\right]. \tag{11}$$

The Einstein equation $G = 8\pi T$ says that w_m is $e^f T(l_+, l_-)$ averaged over the 2-surface. It seems that w_j represents effective energy density due to deviation from spherical symmetry (eg. angular momentum).

The term wAr' should be a work term done along the horizon. For example, for an electromagnetic field, the term $w_m Ar'$ reduces to the electromagnetic work done along the horizon [8]. It seems that the term $w_j Ar'$ is a work associated with deviation from spherical symmetry (eg. angular momentum) of the trapping horizon.

In this paper the first law of black hole dynamics has been derived without assuming any symmetry or any asymptotic condition. In the derivation we have given a new definition of dynamical surface gravity. In spherical symmetry it reduces to that defined in Ref. [8].

Now some comments are in order. First, besides the first law derived in this paper, there exist the second law [9] and perhaps a third law [15] for the trapping horizon (or apparent horizon). It seems that by using these laws we can formulate black hole thermodynamics consistently as trapping horizon dynamics. However, for this purpose, there is an important open question: we have to associate temperature of quantum fields with the trapping horizon. All we can say here is that the temperature may be given by $\hbar\kappa/2\pi$, where κ is the surface gravity introduced in this paper.

The second and the third (final) comments are on the definition (5) of the surface gravity. By definition, $r\kappa$ is not a local quantity but a quasi-local quantity defined as an integral over the surface $S(\xi^+,\xi^-)$. However, it is expected that the integrand is constant over the integration surface under certain conditions of equilibrium. Although such a statement has not been established, it may be called the zeroth law of black hole dynamics if it is proved in some situation.

A final comment is in order. The surface gravity $\kappa(\xi^+, \xi^-)$ is an invariant of a double-null foliation at the surface. Since a non-null trapping horizon locally determines a unique double-null foliation, the surface gravity is an invariant of the trapping horizon if the horizon is not null. On the other hand, the null case is ambiguous because of the freedom to rescale the other null direction. Fixing this would require some kind of limiting argument that might be effectively a zeroth law. Therefore, we have to impose an auxiliary condition for the surface gravity $\kappa(\xi^+, \xi^-)$ to work well when the trapping horizon is null. Since surface gravity seems to be related to temperature of quantum fields as stated above, it will be valuable to investigate the auxiliary condition in detail.

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